Received: *November 18, 1999*, Accepted: *January 13, 2000*

# On Affleck-Dine-Seiberg superpotential and magnetic monopoles in supersymmetric QCD

**N. Michael Davies and Valentin V. Khoze**

*Department of Physics, University of Durham*
*Durham, DH1 3LE, UK*
*E-mail:* n.m.davies@durham.ac.uk, valya.khoze@durham.ac.uk

Abstract: Certain exact results in supersymmetric gauge theories are generated by non-perturbative effects different from instantons. In supersymmetric QCD with $N$ colours and $N_{\mathrm{F}}$ fundamental flavours we examine the Affleck-Dine-Seiberg (ADS) superpotential using controlled semi-classical analysis. We show how for $N_{\mathrm{F}} < N-1$ the ADS superpotential arises from monopole contributions to the path integral of the supersymmetric gauge theory compactified on $\mathbb{R}^3 \times S^1$. These are the monopole effects leading to gaugino condensation and confinement of the low-energy $\mathrm{SU}(N-N_{\mathrm{F}})$ supersymmetric gauge theory.

Keywords: Solitons Monopoles and Instantons, Supersymmetry and Duality, Supersymmetric Effective Theories.

## Contents





# 1. Introduction and motivation

One of the famous exact results in supersymmetry is the Affleck-Dine-Seiberg (ADS) superpotential [1] generated in supersymmetric QCD with $N_{\mathrm{F}}$ fundamental flavours and gauge group $\mathrm{SU}(N)$

$$\mathcal{W}_{\mathrm{ADS}}^{N_{\mathrm{F}},N} = (N - N_{\mathrm{F}}) \left( \frac{\Lambda_{N_{\mathrm{F}},N}^{3N-N_{\mathrm{F}}}}{\det_{N_{\mathrm{F}}} (\tilde{Q}^r Q_s)} \right)^{\frac{1}{N-N_{\mathrm{F}}}} . \tag{1.1}$$

It is well known that for $N_{\mathrm{F}} < N-1$ the ADS superpotential receives no contributions from instantons, and arises from non-perturbative effects associated with gaugino condensation in the unbroken $\mathrm{SU}(N - N_{\mathrm{F}})$ gauge group [1]. In our earlier work [2] we showed that gaugino condensation is generated by monopole contributions to the path integral of supersymmetric gauge theories compactified on $\mathbb{R}^3 \times S^1$. In this paper we make an obvious link between these two ideas [1, 2] and explicitly derive the ADS superpotential in the background of $N - N_{\mathrm{F}}$ varieties of $\mathrm{SU}(N - N_{\mathrm{F}})$ monopoles embedded in the $\mathrm{SU}(N)$ supersymmetric QCD with $N_{\mathrm{F}}$ flavours. We will also explain why this semi-classical derivation is valid in the a priori strongly coupled theory. For related work on $\mathbb{R}^3 \times S^1$ in a different context see [3, 4, 5].

The last few years have seen an impressive agreement between known exact results in supersymmetric gauge theories and direct field-theoretical calculations. In most of the cases considered, the non-perturbative information contained in the exact results [6, 7, 8] was restricted to multi-instanton contributions [9]–[13]. At the same time, some of the known exact results in supersymmetric gauge theories are generated by non-perturbative effects different from instantons. We expect that these non-perturbative effects can still be evaluated and understood semi-classically, but in slightly unusual settings. The motivation of this letter and of its predecessor ref. [2] is to show how this works for two important examples: gaugino condensation and the ADS superpotential. In both cases there is a non-abelian, asymptotically free (sub)group which makes the theory strongly coupled. This is an obstacle for a direct application of semi-classical analysis, which makes these cases qualitatively different from the weakly coupled scenarios:

1. In the Seiberg-Witten case [6] the $\mathrm{SU}(N)$ gauge group is broken by the Higgs mechanism to the abelian subgroup $\mathrm{U}(1)^{N-1}$ with effective couplings determined by the VEVs. The holomorphic nature of this dependence, guaranteed by non-renormalization theorems in $\mathcal{N}=2$, then allows one to travel smoothly to an arbitrarily weak coupling regime saturated by the relevant semi-classical physics: perturbation theory, and multi-instanton effects calculated in [9, 10, 11, 12].

2. In general, when there is no unbroken non-abelian subgroup, the F-terms in any $\mathcal{N}=1$ supersymmetric theory can be usually determined with a constrained instanton calculation [1, 17, 18, 19, 10] as reviewed in [20]. A simple example of this set-up is the supersymmetric QCD with $N_{\mathrm{F}} = N-1$.

3. In the AdS/CFT programme [7] the non-abelian gauge group is unbroken, but the $\beta$-function is zero, and for successful multi-instanton calculations [8, 13, 14, 15, 16], the constant coupling can be taken as small as needed.

It has been suspected for a long time [21] that in the strongly coupled theories, instantons should be thought of as composite states of more basic configurations, the 'instanton partons'. These partons would give important contributions to the non-perturbative dynamics at strong coupling. In ref. [2] we identified the instanton partons with the $N$ varieties of monopoles which appear when Minkowski space-time of $\mathrm{SU}(N)$ supersymmetric gauge theory is partially compatified on $\mathbb{R}^3 \times S^1$. This compactification is required for two reasons. First, because of the finite radius $\beta$ of $S^1$, the monopole solutions have finite action, which makes them important semi-classically. Second, the gauge fields with Lorentz index referring to the $S^1$ direction develop vacuum expectation values [2]

$$\langle A_4 \rangle = \mathrm{diag}\left(\frac{N-1}{N}\frac{\pi}{i\beta}, \frac{N-3}{N}\frac{\pi}{i\beta}, \ldots, -\frac{N-1}{N}\frac{\pi}{i\beta}\right), \tag{1.2}$$

which break the non-abelian $\mathrm{SU}(N)$ to the abelian subgroup $\mathrm{U}(1)^{N-1}$. As the VEVs are inversely proportional to the radius $\beta$, the theory becomes weakly coupled at small $\beta$ and can be analysed semi-classically. To return to the strongly coupled theory in Minkowski space, we need to consider the opposite limit of large $\beta$. Since all the F-terms are holomorphic functions of the fields and since the VEVs of the fields (1.2) are holomorphic functions of $\beta$, the power of holomorphy [22] allows to analytically continue the semi-classical values of the F-terms to the strong-coupling regime.

In supersymmetric QCD with $N$ colours and $N_{\rm F}$ fundamental flavours in the Higgs phase, the gauge group is broken by the matter VEVs to $\mathrm{SU}(N-N_{\rm F})$. In section 2 we will review the monopole calculus in the theory with $N_{\rm F}=0$ on $\mathbb{R}^3\times S^1$. The monopole effects in this theory generate the gaugino condensate and give a mass to the dual (magnetic) photon which implies confinement of electric charges. In section 3 we will include the matter fields and start in uncompactified Minkowski space. We will first integrate out the classically massive (higgsed) fields. The resulting effective lagrangean will contain the massless mesons $M=\tilde{Q}Q$ and the strongly-interacting massless $\mathrm{SU}(N-N_{\rm F})$ gauge supermultiplet, $W_{\mathrm{SU}(N-N_{\rm F})}$. These two sectors will be coupled to each other in the effective lagrangean via non-renormalizable interactions generated perturbatively by integrating out massive fields propagating in the loops. The next step will be to integrate out the gauge supermultiplet, $W_{\mathrm{SU}(N-N_{\rm F})}$. This is achieved in the monopole background in the $\mathrm{SU}(N-N_{\rm F})$ theory compactified on $\mathbb{R}^3\times S^1$. The superpotential for mesons — the ADS superpotential (1.1) — arises from the gauge-meson interactions when the gauge supermultiplet is integrated out. Hence, the ADS superpotential (1.1) is a combination of perturbative and monopole effects. The analysis of section 3 can be schematically represented as

$$\begin{aligned}\int \mathcal{D}W_{\mathrm{SU}(N)}\,\mathcal{D}Q\,\mathcal{D}\tilde{Q}\,e^{i\int d^4x\,\mathcal{L}(W_{\mathrm{SU}(N)},Q,\tilde{Q})} &= \int \mathcal{D}W_{\mathrm{SU}(N-N_{\rm F})}\,\mathcal{D}M\,e^{i\int d^4x\,\mathcal{L}^{(1)}_{\rm eff}(W_{\mathrm{SU}(N-N_{\rm F})},M)}\\ &= \int \mathcal{D}M\,e^{i\int d^4x\,\mathcal{L}^{(2)}_{\rm eff}(M)}\,. \end{aligned}\tag{1.3}$$

We stress that eq. (1.3) represents a *direct* evaluation of the ADS superpotential contained in $\mathcal{L}^{(2)}_{\rm eff}(M)$. In particular we are not apealing to renormalization group arguments relating expressions for different values of $N_{\rm F}$. None of the meson degrees of freedom were integrated out on the right hand side of eq. (1.3), and the F-term of $\mathcal{L}^{(2)}_{\rm eff}(M)$ is the definition of the general ADS superpotential $\mathcal{W}^{N_{\rm F},N}_{\rm ADS}$.

## 2. Monopoles in SYM on $\mathbb{R}^3\times S^1$

In this section we consider the pure $\mathcal{N}=1$ supersymmetric Yang-Mills theory coupled to a background chiral superfield $T^{(\mu)}$

$$\mathcal{Z}(T^{(\mu)}) = \int \mathcal{D}W\,e^{i\int d^4x\,\mathcal{L}(W,T^{(\mu)})} = e^{-i\left(\int d^4x\,d^2\theta\,\mathcal{W}_{\rm eff}(T^{(\mu)})+\mathrm{h.c.}\right)}\,. \tag{2.1}$$

Here $W$ denotes the $\mathrm{SU}(N)$ field-strength chiral superfield, and $T^{(\mu)}$ is a non-dynamical background chiral superfield

$$T^{(\mu)}(y,\theta)\equiv\tau(y)+\sqrt{2}\,\theta^\alpha\chi^\tau_\alpha(y)+\theta^2F^\tau(y)\,,\qquad \langle\tau\rangle=\frac{4\pi i}{g^2(\mu)}+\frac{\theta(\mu)}{2\pi} \tag{2.2}$$

The superscript $\mu$ indicates the (high) energy scale where $T^{(\mu)}$ becomes dynamical. The microscopic lagrangean defining the theory reads

$$\mathcal{L} = \frac{1}{4\pi}\,\mathrm{Im}\,\mathrm{tr}_N \int d^2\theta\, T^{(\mu)}\, W^\alpha W_\alpha\,. \tag{2.3}$$

The dynamical quantities we are after are the effective superpotential $\mathcal{W}_{\text{eff}}(T^{(\mu)})$ defined in eq. (2.1) and the gaugino condensate $\langle \mathrm{tr}_N \lambda^2\rangle$ in the presence of the $\tau$ field. The two quantities are related due to a functional identity

$$\left\langle \mathrm{tr}_N \lambda^2 \right\rangle = \frac{8\pi}{\mathcal{Z}(T^{(\mu)})}\frac{\delta}{\delta F^\tau(x)}\,\mathcal{Z}(T^{(\mu)})\Big|_{T^{(\mu)}(y,\theta)=\tau} = -8\pi i\,\frac{\partial \mathcal{W}_{\text{eff}}(\tau)}{\partial\tau}\,, \tag{2.4}$$

which trivially follows from eqs. (2.1)-(2.3).

**Gauge theory on $\mathbb{R}^3 \times S^1$** Let $x_4$ be periodic with period $\beta/2\pi$.[1] We must then impose periodic boundary conditions for bosons, $A_m(x_\mu, x_4 = 0) = A_m(x_\mu, x_4 = \beta)$ and fermions, $\lambda(x_\mu, x_4 = 0) = \lambda(x_\mu, x_4 = \beta)$ to preserve supersymmetry. In addition, the local gauge group itself must also be composed of gauge transformations periodic on $S^1$: $\mathrm{U}(x_\mu, x_4 = 0) = \mathrm{U}(x_\mu, x_4 = \beta)$. We now summarize the results of the analysis carried out in ref. [2].

1. There are $N$ classically flat directions of the vacuum moduli space of the compactified theory parametrized by $\langle A_4\rangle = \mathrm{diag}(a_1, a_2, \ldots, a_N)$. This classical moduli space is lifted non-trivially at the quantum level to

$$\langle A_4\rangle = \mathrm{diag}\left(\frac{N-1}{N}\frac{\pi}{i\beta}, \frac{N-3}{N}\frac{\pi}{i\beta}, \ldots, -\frac{N-1}{N}\frac{\pi}{i\beta}\right), \tag{2.5}$$

   by the monopole-generated superpotential derived in [2, section 3]. The distinctive feature of (2.5) is the constant equal spacing between the VEVs $a_j$.

2. The semi-classical physics of the $\mathbb{R}^3 \times S^1$ SU($N$) theory is described by configurations of monopoles of $N$ different types. In general, the field configurations of the $\mathbb{R}^3 \times S^1$ theory which are relevant in the semi-classical regime are both instantons and monopoles. Remarkably, the instantons on $\mathbb{R}^3 \times S^1$ can be understood as composite configurations of $N$ single monopoles, one of each of the $N$ different types [23, 24, 25]. One expects in an SU($N$) theory on $\mathbb{R}^4$ that the lowest charged monopoles come in $N-1$ different varieties, carrying a unit of magnetic charge from each of the U(1) factors of the $\mathrm{U}(1)^{N-1}$ gauge group left unbroken by the VEVs. The additional monopole, needed to make up the $N$ types, is specific to the compactification on $\mathbb{R}^3 \times S^1$ [24, 25]. The magnetic charge of the new monopole is such that when all $N$ types of monopoles are present, the magnetic charges cancel and the resulting configuration only carries a unit instanton charge.

[1] The indices run over $m = 1, 2, 3, 4$ and $\mu = 1, 2, 3$. Our conventions are the same as in ref. [2].

**Monopole calculus in** SU(2) The standard BPS monopole solution in Hedgehog gauge [26] is

$$A_4^{\text{BPS}}(x_\nu) = (v|x| \coth(v|x|) - 1)\frac{x_a}{|x|^2}\frac{\tau^a}{2i}\,,$$
$$A_\mu^{\text{BPS}}(x_\nu) = \Big(1 - \frac{v|x|}{\sinh(v|x|)}\Big)\epsilon_{\mu\nu a}\frac{x_\nu}{|x|^2}\frac{\tau^a}{2i}\,. \tag{2.6}$$

These expressions are obviously independent of the $S^1$ variable $x_4$, since the latter can be thought of as the time coordinate of the static monopole. The boundary values of (2.6) as $|x| \to \infty$, when gauge rotated to unitary (singular) gauge, agree with (2.5) for $v = \pi/\beta$

$$A_4^{\text{BPS}} \;\to\; v\frac{\tau^3}{2i} = \frac{\pi}{\beta}\frac{\tau^3}{2i} = \langle A_4\rangle\,. \tag{2.7}$$

The monopole solution (2.6) satisfies the self-duality equations and has topological charge

$$Q \equiv \frac{1}{16\pi^2}\int_0^\beta dx_4 \int d^3x\, \mathrm{tr}\, {}^*F_{mn}F^{mn} = \frac{\beta v}{2\pi} = \frac{1}{2}\,. \tag{2.8}$$

The monopole has magnetic charge one, instanton charge zero, and the action $S_{\text{BPS}}$ is

$$S_{\text{BPS}} = \frac{4\pi}{g^2(\mu)}\beta v = \frac{4\pi^2}{g^2(\mu)}\,, \qquad S_{\text{BPS}}(T^{(\mu)}) = -i\pi T^{(\mu)}\,. \tag{2.9}$$

The solution (2.6) precisely two adjoint fermion zero modes $\lambda_\alpha^{\text{BPS}} = \frac{1}{2}\xi_\beta(\sigma^m\bar\sigma^n)_\alpha^{\ \beta}F_{mn}^{\text{BPS}}$, with normalization [27]:

$$\int d^3X \int d^2\xi\, \mathrm{tr}\,(\lambda^{\text{BPS}\,\alpha}(x)\lambda_\alpha^{\text{BPS}}(x)) = \frac{2g^2 S_{\text{BPS}}}{\beta} = \frac{8\pi^2}{\beta}\,. \tag{2.10}$$

Here $\sigma^m$ and $\bar\sigma^n$ are the four Pauli matrices and $\xi_\beta$ is the two-component Grassmann collective coordinate. The semi-classical integration measure of the standard single-monopole on $\mathbb{R}^3 \times S^1$ reads [27]:

$$\int d\mu^{\text{BPS}} = \mu^3\, e^{-S_{\text{BPS}}(T^{(\mu)})}\int \frac{d^3X}{(2\pi)^{3/2}}\, 8\pi^3 \int_0^{2\pi}\frac{d\Omega}{\sqrt{2\pi}}\frac{2\pi}{v}\int d^2\xi \frac{1}{8\pi^2}\,. \tag{2.11}$$

This measure is obtained in the standard way by changing variables in the path integral from field-fluctuations around the monopole to the monopole's collective coordinates. The UV-regularized determinants over non-zero eigenvalues of the quadratic fluctuation operators cancel between fermions and bosons due to supersymmetry. The ultra-violet divergences are regularized in the Pauli-Villars scheme, which explains the appearance of the Pauli-Villars mass scale $\mu$. We can now compute the single monopole contribution to $\langle \mathrm{tr}\lambda^2\rangle$ as in ref. [2]:

$$\big\langle \mathrm{tr}\lambda^2\big\rangle_{\text{BPS}} = \int d\mu^{\text{BPS}}\, \mathrm{tr}\,(\lambda^{\text{BPS}\,\alpha}(x)\lambda_\alpha^{\text{BPS}}(x)) = 8\pi^2\mu^3 \exp\left[i\pi T^{(\mu)}\right]. \tag{2.12}$$

As explained in ref. [2] the monopole of the second type will give a contribution identical to (2.12). Putting the two together we get

$$\left\langle \mathrm{tr}\lambda^2 \right\rangle = 16\pi^2 \,\mu^3 \exp\left[i\pi T^{(\mu)}\right]. \tag{2.13}$$

Substituting (2.13) into (2.4) gives the equation for the superpotential $\mathcal{W}_{\rm eff}(\tau)$ with the solution

$$\mathcal{W}_{\rm eff}(T^{(\mu)}) = 2\mu^3 \,\exp\left[\pi i T^{(\mu)}\right]. \tag{2.14}$$

The expressions (2.13) and (2.14) are, in spite of appearences, renormalization group invariant and $\mu$-independet. We also note that the dependence on the $S^1$ radius $\beta$ disappeared, and the decompactification limit $\beta \to \infty$ does not change the final results (2.13) and (2.14).

**Generalization to** SU$(N)$ The calculation of the gaugino condensate and the superpotential can be straightforwardly generalized to the case of gauge group SU$(N)$. The quantum vacuum has

$$a_j - a_{j+1} = \frac{2\pi}{iN\beta} \;\mathrm{mod}\; \frac{2\pi}{i\beta}\,, \qquad j = 1, 2, \ldots, N\,, \tag{2.15}$$

and each of the $N$ types of monopoles has equal actions and equal topological charges:

$$S_{\rm mono} = \frac{8\pi^2}{Ng^2(\mu)}\,, \qquad S_{\rm mono}(T^{(\mu)}) = -\frac{2\pi i}{N}\,T^{(\mu)}\,, \qquad Q_{\rm mono} = \frac{1}{N}\,. \tag{2.16}$$

The contribution of $N$ monopoles to the gaugino condensate and the superpotential is straightforward:

$$\left\langle \mathrm{tr}\lambda^2 \right\rangle = 16\pi^2\mu^3 \exp\left[\frac{2\pi i}{N}T^{(\mu)}\right], \qquad \mathcal{W}_{\rm eff}(T^{(\mu)}) = N\mu^3\; \exp\left[\frac{2\pi i}{N}T^{(\mu)}\right]. \tag{2.17}$$

This concludes our analysis of supersymmetric QCD with $N_{\rm F} = 0$.

## 3. ADS superpotential in supersymmetric QCD

The microscopic theory is defined in terms of the SU$(N)$ vector superfield coupled to $N_{\rm F}$ chiral superfields in the $[N]$ reperesentation, $Q_{ur}$ ($u = 1, \ldots, N;\; r = 1, \ldots, N_{\rm F}$), and to $N_{\rm F}$ chiral superfields in the $[\bar{N}]$ reperesentation, $\tilde{Q}^{ru}$. The global classical symmetry of the lagrangean is

$$\mathrm{SU}(N_{\rm F})_{\rm left} \times \mathrm{SU}(N_{\rm F})_{\rm right} \times \mathrm{U}(1)_{\rm V} \times \mathrm{U}(1)_{\rm A} \times \mathrm{U}(1)_{\rm R}\,, \tag{3.1}$$

where $\mathrm{U}(1)_{\rm R}$ is the anomaly-free combination of the R-symmetry and the axial $\mathrm{U}(1)_{\rm A}$:

$$\begin{aligned} W(\theta) &\to e^{-i\alpha} W(e^{i\alpha}\theta)\,, \\ Q(\theta) &\to e^{i\alpha(N-N_{\rm F})/N_{\rm F}} Q(e^{i\alpha}\theta)\,, \\ \tilde{Q}(\theta) &\to e^{i\alpha(N-N_{\rm F})/N_{\rm F}} \tilde{Q}(e^{i\alpha}\theta)\,. \end{aligned} \tag{3.2}$$

At the quantum level the classical $U(1)_A$ symmetry is anomalous, and the global *quantum* symmetry of the lagrangean is $SU(N_F)_{left} \times SU(N_F)_{right} \times U(1)_V \times U(1)_R$.

The classical vacuum state is determined in the standard way via the D-flatness condition and can be brought to a simple 'rectangular diagonal' form:

$$\langle A_{ur} \rangle = \begin{cases} \delta_{ur} v_r \,, & u = 1, \ldots, N_F \\ 0 \,, & u = N_F + 1, \ldots, N \end{cases} = \langle \tilde{A}^\dagger_{ur} \rangle \,. \tag{3.3}$$

The $N_F$ complex vacuum expectation values $v_1, \ldots, v_{N_F}$ are not fixed by the classical lagrangean, and parameterize the $N_F$-complex-dimensional classical vacuum moduli space of the theory. It is now straightforward to determine which symmetries of the lagrangean are left unbroken by the classical vacuum (3.3). The $SU(N)$ gauge symmetry is spontaneously broken by (3.3) to $SU(N - N_F)$. In order to determine the surviving global symmetry it is convenient to restrict ourselves to the case of all VEVs equal, $v_1 = v_2 = \ldots \equiv v$. This choice will not affect the counting of classically massless degrees of freedom, but will simplify the symmetry reasoning. This vacuum state breaks the $U(1)_A$ and $U(1)_R$. It also breaks $SU(N_F)_{left} \times SU(N_F)_{right}$ to the diagonal $SU(N_F)$ subgroup that rotates $A_r$ and $\tilde{A}^\dagger_r$ in the same way and is further compensated by the gauge transformation, such that the vacuum is unchanged. Thus, the global symmetry respected by the vacuum is $SU(N_F) \times U(1)_V$. Hence, we expect ${N_F}^2 + 1$ massless Goldstone bosons at the perturbative level, i.e. ${N_F}^2 + 1$ real massless scalars coming from the broken generators of the global $SU(N_F) \times U(1)_A \times U(1)_R$. When the non-perturbative effects are taken into account one of these massless degrees of freedom will acquire a mass due to the $U(1)_A$ anomaly, and ${N_F}^2$ real scalar degrees of freedom will remain massless.

At the same time, the number of *classically massless* real scalar degrees of freedom is $2{N_F}^2$. This is twice the number of exact Goldstone bosons. The remaining ${N_F}^2$ classically massless real scalars must be lifted by a non-perturbatively generated superpotential. The functional form of this superpotential eq. (1.1) was uniquely determined in [1].

The ADS superpotential appears in the low-energy effective description of the microscopic theory with $N_F \leq N - 1$. It is important to distinguish between the two cases: $N_F = N - 1$ where the gauge group is completely broken by the vacuum; and $N_F < N - 1$ where the non-abelian gauge subgroup $SU(N - N_F)$ is unbroken. The first case is relatively simple and is well understood [10, 19, 1]. For $N_F = N - 1$ the superpotential is generated at the 1-instanton level, and since the gauge group is completely broken, the instanton calculation is reliable and infra-red safe. In the second, more general case, $N_F < N - 1$, instantons are known to give trivially vanishing conributions to (1.1), nevertheless the renormalization group decoupling argument from $N_F = N - 1$ to $N_F < N - 1$ requires the superpotential to be non-vanishing, and determines the normalzation consant in (1.1) [22, 10, 19] to be $N - N_F$.



JHEP01(2000)015

$N_{\rm F} < N-1$**: step one.** For $N_{\rm F} < N-1$ there is an unbroken gauge subgroup $\mathrm{SU}(N-N_{\rm F})$ and as discussed in the Introduction, it is important to realise that the generation of the ADS superpotential is the two-step process represented by eq. (1.3). We first look at the perturbative decoupling of the massive vector bosons and Higgs bosons. This step is the generalization of the Affleck-Dine-Seiberg $N_{\rm F}=1$ argument from ref. [1, section 4] to all $N_{\rm F} < N-1$. The relevant matter degrees of freedom are the $2{N_{\rm F}}^2$ classically massless real scalars and their superpartners. They can be packaged into ${N_{\rm F}}^2$ chiral superfield (complex) degrees of freedom. In fact, there are precisely ${N_{\rm F}}^2$ gauge-invariant composite chiral meson superfields $M^r_s(x,\theta) = \tilde{Q}^{ru}(x,\theta)Q_{us}(x,\theta)$. As the scalar components of the superfields $M$ have the mass-dimension two we prefer to use an equivalent parametrisation of matter degrees of freedom in terms of mass-dimension one chiral superfields

$$\Phi^r_s = \frac{v}{\sqrt{2}}\log\frac{\tilde{Q}^r Q_s}{v^2}\,. \tag{3.4}$$

The logarithm is used for the later convenience and the mass-dimension one in (3.4) is achieved via explicit dependence on the classical VEV $v$ which will be necessary in the perturbative decoupling treatment, but will disappear from the final results. In fact, all the non-renormalizable interactions induced by integrating out the heavy fields will be in terms of higher dimensional operators of the light fields divided by the powers of $v$. A perturbative (or semi-classical) treatment is applicable as long as $v$ is large compared to $\Lambda_{N_{\rm F},N}$, which will be assumed. There is a unique operator of dimension five which couples the two sectors,

$$\Gamma = \frac{\sqrt{2}}{32\pi^2}\frac{1}{v}\int d^2\theta(\mathrm{Tr}_{N_{\rm F}}\Phi)W^{a\alpha}W^a_\alpha + \text{h.c.}\,. \tag{3.5}$$

The the normalization factor $\frac{\sqrt{2}}{32\pi^2}$ can be determined either from the corresponding 1-loop perturbative supergraphs, or from requiring the $\mathrm{U}(1)_{\rm R}$ invariance of the effective lagrangean

$$\begin{aligned}\mathcal{L}^{(1)}_{\rm eff} = &\sum_{a=1}^{(N-N_{\rm F})^2-1}\int d^2\theta\left(\frac{1}{4g^2(\mu)}W^{a\alpha}W^a_\alpha + \frac{\sqrt{2}}{32\pi^2}\frac{1}{v}(\mathrm{Tr}_{N_{\rm F}}\Phi)W^{a\alpha}W^a_\alpha\right) + \text{h.c.} + \\ &+\int d^2\theta d^2\bar{\theta}\Phi^\dagger\Phi\,.\end{aligned} \tag{3.6}$$

Under $\mathrm{U}(1)_{\rm R}$ the pure gauge term is anomalous

$$\delta\,\frac{1}{4g^2}\int d^2\theta\, W^2 = -\frac{(N-N_{\rm F})}{16\pi^2}F^{mn*}F_{mn}\,, \tag{3.7}$$

and the $\mathrm{U}(1)_{\rm R}$ transformation of $\Phi$, read from (3.2),

$$\delta\,\Phi^r_s = iv\sqrt{2}\frac{(N-N_{\rm F})}{N_{\rm F}} \tag{3.8}$$

leaves (3.6) invariant as required.

There are of course even higher-dimensional operators coupling the two sectors, but they are suppressed by higher powers of $v$ and will not contribute at the relevant order. Furthermore, perturbative non-renormalization theorems will prevent the generation of the superpotential made solely of the $\Phi$ fields as no such superpotential existed classically. Finally, note that eq. (3.6) should be thought of as the Wilson effective lagrangean with $\mu$ in $g^2(\mu)$ on the right hand side of (3.6) being the Wilson scale. In deriving $\mathcal{L}^{(1)}_{\text{eff}}$ we have integrated out all the degrees of freedom with masses and virtualities greater than $\mu$, where $\mu \le v$.

$N_{\rm F} < N-1$**: step two.** The gauge sector together with the gauge–meson interactions sector of the effective lagrangean $\mathcal{L}^{(1)}_{\text{eff}}$ can be represented as

$$\mathcal{L}^{\text{gauge}}_{\text{eff}} = \frac{1}{8\pi}\,\text{Im} \sum_{a=1}^{(N-N_{\rm F})^2-1} \int d^2\theta\, T^{(\mu)}\, W^{a\alpha} W^a_\alpha\,, \tag{3.9}$$

where[2] we have introduced a chiral superfield

$$T^{(\mu)} = \frac{4\pi i}{g^2(\mu)} + \frac{1}{2\pi}\frac{i\sqrt{2}\,\text{Tr}_{N_{\rm F}}\Phi}{v}\,. \tag{3.10}$$

We now see from equations (1.3) and (3.9) that the problem of deriving the ADS superpotential in the theory with $N$ colours and $N_{\rm F}$ flavours is reduced to integrating out the vector supermultiplet $W_{\text{SU}(N-N_{\rm F})}$ in the pure $\text{SU}(N-N_{\rm F})$ supersymmetric Yang-Mills theory coupled to the background superfield $T^{(\mu)}$

$$\int \mathcal{D}W_{\text{SU}(N-N_{\rm F})}\, e^{i\int d^4x\, \mathcal{L}^{\text{gauge}}_{\text{eff}}(W_{\text{SU}(N-N_{\rm F})},T^{(\mu)})} = e^{-i\left(\int d^4x\, d^2\theta\, \mathcal{W}_{\text{eff}}(T^{(\mu)}) + \text{h.c.}\right)}\,. \tag{3.11}$$

Applying the analysis of section 2 we conclude that the functional integral over the gauge supermultiplet receives contributions from $N-N_{\rm F}$ varieties of monopoles which arise in the $\text{SU}(N-N_{\rm F})$ supersymmetric Yang-Mills theory partially compactified on $\mathbb{R}^3\times S^1$. The semi-classical evaluation of the integral in the monopole background is exact and in the decompactification limit $\mathbb{R}^3\times S^1 \to \mathbb{R}^4$ gives, in analogy with eq. (2.17)

$$\mathcal{W}_{\text{eff}}(T^{(\mu)}) = (N-N_{\rm F})\,\mu^3\,\exp\left[\frac{2\pi i}{N-N_{\rm F}}T^{(\mu)}\right]. \tag{3.12}$$

In the rest of this section we want to demonstrate that the superpotential $\mathcal{W}_{\text{eff}}(T^{(\mu)})$, given by eq. (3.12) is in fact equal to the ADS superpotential $\mathcal{W}_{\text{ADS}}$ (1.1). Indeed, substitute the definition of $T^{(\mu)}$, eq. (3.10) into eq. (3.12),

$$\mathcal{W}_{\text{eff}}(T^{(\mu)}) = (N-N_{\rm F})\mu^3 e^{-\frac{8\pi^2}{g^2(\mu)}\frac{1}{N-N_{\rm F}}} \left(\frac{v^{2N_{\rm F}}}{\det_{N_{\rm F}}(\tilde{Q}^r Q_s)}\right)^{\frac{1}{N-N_{\rm F}}}. \tag{3.13}$$

[2] $\mathcal{L}^{\text{gauge}}_{\text{eff}}$ is defined as the first line on the right hand side of eq. (3.6), i.e. $\mathcal{L}^{(1)}_{\text{eff}}$ minus the pure matter sector.

Now note that the $\mu$-dependent terms combine to the RG invariant definition of the dynamical scale of the supersymmetric pure $\mathrm{SU}(N - N_{\mathrm{F}})$ gauge theory

$$\mu^3 \, e^{-\frac{8\pi^2}{g^2(\mu)} \frac{1}{N-N_{\mathrm{F}}}} = \Lambda^3_{0,N-N_{\mathrm{F}}} \,. \tag{3.14}$$

Furthermore, the decoupling relation, $\Lambda^{3N-N_{\mathrm{F}}}_{N_{\mathrm{F}},N} = v^{2N_{\mathrm{F}}} \Lambda^{3(N-N_{\mathrm{F}})}_{0,N-N_{\mathrm{F}}}$ enables us to rewrite the final result in terms of the dynamical scale $\Lambda_{N_{\mathrm{F}},N}$ of the original SQCD:

$$\mathcal{W}_{\mathrm{eff}} = (N - N_{\mathrm{F}}) \left( \frac{\Lambda^{3N-N_{\mathrm{F}}}_{N_{\mathrm{F}},N}}{\det_{N_{\mathrm{F}}} (\tilde{Q}^r Q_s)} \right)^{\frac{1}{N-N_{\mathrm{F}}}} = \mathcal{W}^{N_{\mathrm{F}},N}_{\mathrm{ADS}} \,. \tag{3.15}$$

## Acknowledgments

We thank Diego Bellisai and Tim Hollowood for valuable discussions. N.M.D. acknowledges a PPARC studentship.



## References

[1] I. Affleck, M. Dine and N. Seiberg, *Dynamical supersymmetry breaking in supersymmetric QCD*, *Nucl. Phys.* **B 241** (1984) 493.

[2] N.M. Davies, T.J. Hollowood, V.V. Khoze and M.P. Mattis, *Gluino condensate and magnetic monopoles in supersymmetric gluodynamics*, `hep-th/9905015`.

[3] N. Seiberg and E. Witten, *Gauge dynamics and compactification to three dimensions*, in *The mathematical beauty of physics*, J.M. Drouffe and J.-B. Zubered., p. 333, World Scientific 1997, [`hep-th/9607163`].

[4] O. Aharony, A. Hanany, K. Intriligator, N. Seiberg and M.J. Strassler, *Aspects of $N = 2$ supersymmetric gauge theories in three dimensions*, *Nucl. Phys.* **B 499** (1997) 67 [`hep-th/9703110`].

[5] N. Dorey, *An elliptic superpotential for softly broken $N = 4$ supersymmetric Yang-Mills theory*, *J. High Energy Phys.* **07** (1999) 021 [`hep-th/9906011`].

[6] N. Seiberg and E. Witten, *Electric-magnetic duality, monopole condensation and confinement in $N = 2$ supersymmetric Yang-Mills theory*, *Nucl. Phys.* **B 426** (1994) 19 [`hep-th/9407087`].

[7] J. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [`hep-th/9711200`];
S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from non-critical string theory*, *Phys. Lett.* **B 428** (1998) 105 [`hep-th/9802109`];
E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [`hep-th/9802150`].

[8] T. Banks and M.B. Green, *Non-perturbative effects in $AdS_5 \times S^5$ string theory and $D = 4$ SUSY Yang-Mills*, *J. High Energy Phys.* **05** (1998) 002 [hep-th/9804170];
M. Bianchi, M.B. Green, S. Kovacs and G. Rossi, *Instantons in supersymmetric Yang-Mills and D-instantons in IIB superstring theory*, *J. High Energy Phys.* **08** (1998) 013 [hep-th/9807033];
M.B. Green, *Interconnections between type-II superstrings, M-theory and $N = 4$ Yang-Mills*, hep-th/9903124.

[9] N. Dorey, V.V. Khoze and M.P. Mattis, *Multi-instanton calculus in $N = 2$ supersymmetric gauge theory*, *Phys. Rev.* **D 54** (1996) 2921 [hep-th/9603136]; *Multi-instanton calculus in $N = 2$ supersymmetric gauge theory, 2. Coupling to matter*, *Phys. Rev.* **D 54** (1996) 7832 [hep-th/9607202];
H. Aoyama, T. Harano, M. Sato and S. Wada, *Multi-instanton calculus in $N = 2$ supersymmetric QCD*, *Phys. Lett.* **B 388** (1996) 331 [hep-th/9607076].

[10] D. Finnell and P. Pouliot, *Instanton calculations versus exact results in four-dimensional susy gauge theories*, *Nucl. Phys.* **B 453** (1995) 225 [hep-th/9503115].

[11] K. Ito and N. Sasakura, *One-instanton calculations in $n = 2$ supersymmetric* $\mathrm{SU}(n_c)$ *Yang-Mills theory*, *Phys. Lett.* **B 382** (1996) 95 [hep-th/9602073]; *One-instanton calculations in $N = 2$* $\mathrm{SU}(N_c)$*) supersymmetric QCD*, *Mod. Phys. Lett.* **A 12** (1997) 205 [hep-th/9609104].

[12] V.V. Khoze, M.P. Mattis and M.J. Slater, *The instanton hunter's guide to supersymmetric* $\mathrm{SU}(N)$ *gauge theory*, *Nucl. Phys.* **B 536** (1998) 69 [hep-th/9804009].

[13] N. Dorey, T.J. Hollowood, V.V. Khoze, M.P. Mattis and S. Vandoren, *Multi-instanton calculus and the AdS/CFT correspondence in $N = 4$ superconformal field theory*, *Nucl. Phys.* **B 552** (1999) 88 [hep-th/9901128].

[14] T.J. Hollowood and V.V. Khoze, *ADHM and D-instantons in orbifold AdS/CFT duality*, hep-th/9908035;
T.J. Hollowood, V.V. Khoze and M.P. Mattis, *Summing the instanton series in $N = 2$ superconformal large- $n$ QCD*, *J. High Energy Phys.* **10** (1999) 019 [hep-th/9905209].

[15] E. Gava, K.S. Narain and M.H. Sarmadi, *Instantons in $\mathcal{N} = 2$* $\mathrm{Sp}(N)$ *superconformal gauge theories and the AdS/CFT correspondence*, hep-th/9908125;
T.J. Hollowood, *Instantons, finite $\mathcal{N} = 2$* $\mathrm{Sp}(N)$ *theories and the AdS/CFT correspondence*, *J. High Energy Phys.* **11** (1999) 012 [hep-th/9908201].

[16] T.J. Hollowood, V.V. Khoze and M.P. Mattis, *Instantons in $N = 4$* $\mathrm{Sp}(N)$ *and* $\mathrm{SO}(N)$ *theories and the AdS/CFT correspondence*, hep-th/9910118.

[17] V.A. Novikov, M.A. Shifman, A.I. Vainshtein and V.I. Zakharov, *Supersymmetric instanton calculus: gauge theories with matter*, *Nucl. Phys.* **B 260** (1985) 157.

[18] J. Fuchs and M.G. Schmidt, *Instanton induced Green functions in the superfield formalism*, *Z. Physik* **C 30** (1986) 161.

[19] S.F. Cordes, *The instanton induced superpotential in supersymmetric QCD*, *Nucl. Phys.* **B 273** (1986) 629.

[20] M. Shifman and A. Vainshtein, *Instantons versus supersymmetry: fifteen years later*, `hep-th/9902018`.

[21] A.A. Belavin, V.A. Fateev, A.S. Schwarz and Y.S. Tyupkin, *Quantum fluctuations of multi-instanton solutions*, *Phys. Lett.* **B 83** (1979) 317;
V.A. Fateev, I.V. Frolov and A.S. Shvarts, *Quantum fluctuations of instantons in the nonlinear sigma model*, *Nucl. Phys.* **B 154** (1979) 1;
B. Berg and M. Luscher, *Computation of quantum fluctuations around multi-instanton fields from exact Green's functions: the $\mathbb{CP}^{n-1}$ case*, *Comm. Math. Phys.* **69** (1979) 57.

[22] N. Seiberg, *Naturalness versus supersymmetric nonrenormalization theorems*, *Phys. Lett.* **B 318** (1993) 469 [`hep-ph/9309335`];
K. Intriligator and N. Seiberg, *Lectures on supersymmetric gauge theories and electric-magnetic duality*, *Nucl. Phys.* **45BC** *(Proc. Suppl.)* (1996) 1 [`hep-th/9509066`].

[23] W. Nahm, *Self-dual monopoles and calorons*, in *Lecture notes in Physics 201*, G. Denado et. al. eds., 1984, p. 189;
H. Garland and M.K. Murray, *Kac-Moody monopoles and periodic instantons*, *Comm. Math. Phys.* **120** (1988) 335.

[24] K. Lee and P. Yi, *Monopoles and instantons on partially compactified D-branes*, *Phys. Rev.* **D 56** (1997) 3711 [`hep-th/9702107`];
K. Lee, *Instantons and magnetic monopoles on $\mathbb{R}^3 \times S^1$ with arbitrary simple gauge groups*, *Phys. Lett.* **B 426** (1998) 323 [`hep-th/9802012`];
K.Y. Lee and C. hai Lu, SU(2) *calorons and magnetic monopoles*, *Phys. Rev.* **D 58** (1998) 025011 [`hep-th/9802108`].

[25] T.C. Kraan and P. van Baal, *Exact T-duality between calorons and Taub-NUT spaces*, *Phys. Lett.* **B 428** (1998) 268 [`hep-th/9802049`]; *Periodic instantons with non-trivial holonomy*, *Nucl. Phys.* **B 533** (1998) 627 [`hep-th/9805168`]; *Monopole constituents inside* SU($N$) *calorons*, *Phys. Lett.* **B 435** (1998) 389 [`hep-th/9806034`].

[26] E.B. Bogomol'nyi, *Sov. J. Phys.* **24** (1977) 97;
M.K. Prasad and C.M. Sommerfield, *An exact classical solution for the't Hooft monopole and the Julia-Zee dyon*, *Phys. Rev. Lett.* **35** (1975) 760.

[27] N. Dorey, V.V. Khoze, M.P. Mattis, D. Tong and S. Vandoren, *Instantons, three-dimensional gauge theory and the Atiyah-Hitchin manifold*, *Nucl. Phys.* **B 502** (1997) 59 [`hep-th/9703228`].